Sums of Rows of Generalized Pascal Triangles

Sadek Mohammed and Greg Dresden

August 2022

Abstract

Some generalizations of Pascal's triangle are stated along with some formulas - accompanied with their proofs - for the sums of entries in the rows of those generalizations. The paper ends in a formula that gives the row sum for any given triangle.

1 Introduction

Although it is attributed to the french mathematician Blaise Pascal who, in 1665, solved probability problems using known properties of the triangle, Pascal's triangle was invented by a Persian mathematician who was born more than half a millennium before Pascal [3], called Al-Kharaji (953-1029). The triangle has many other inventors (or more precisely contributors) in China and Europe.

Formulas for row sums of Pascal's triangle and its generalizations (also known as Pascalized Triangles) are as diverse as the triangle's inventors. The paper aims to get formulas for the row sums of Pascal's triangle and some of its generalizations. Afterwards, the paper generalizes a formula for any given triangle.

Pascal's triangle can be constructed by putting 1s on the outer diagonals of the triangle and generating the entry on the n-th row and the k-th column, $T_{n,k}$, by using two entries from the previous row according to the following equation:

$$T_{n,k} = T_{n-1,k} + T_{n-1,k-1} \tag{1}$$

Pascal's triangle could be generalized in two main ways:

◊ Substituting 1s on the outer diagonals by Fibonacci numbers or other sets like those in section (2).

◊ Making each entry in any row the result of using more than two entries from the previous row. More formally, substituting Formula (1) by other formulas like those in section (3).

It is well known that the sum of entries on the n-th row of the original Pascal triangle is 2^n. There are two main proofs for those formulas: one comes from the combinatorial interpretation of the triangle [1] and the other from the fact that each entry is the result of the contribution of two entries from the previous row.

Our question is whether we can obtain a formula for any triangle formed by the formula (1) or not, and the answer is clearly yes.

Theorem 1. *For any $n \geq 1$, the sum of entries of the n-th row, $R(n)$, is worked out by the following recursive formula:*

$$R(n) = 2 \cdot R(n-1) + T_{n,0} + T_{n,n} - T_{n-1,n-1} - T_{n-1,0} \tag{2}$$

Proof. $R(n) = \sum_{k=0}^{n} T_{n,k}$, where as equation 1 does not work for entries $T_{n,0}$ and $T_{n,n}$, $R(n)$ could be best written as:

$$R(n) = T_{n,0} + \sum_{k=1}^{n} T_{n,k} + T_{n,n}$$

By substituting with equation 1, the formula becomes:

$$R(n) = T_{n,0} + \sum_{k=1}^{n-1} T_{n-1,k} + \sum_{k=1}^{n-1} T_{n-1,k-1} + T_{n,n}$$

Writing the formula in terms of the sum of entries of the previous row:

$$R(n) = T_{n,0} + R(n) - T_{n-1,0} + R(n) - T_{n-1,n-1} + T_{n,n}$$

Rearranging gives us our target equation (2) ■

Since all the entries on the diagonals of Pascal's triangle are equal and the 0-th row has the sum 2^0, it is obvious why equation (2) results in the 2^n relation.

2 Set-Pascalized Triangles

This section will talk about the first type of Pascalized Triangles, which are triangles formed by replacing 1s on Pascal's Triangle by other sets. The first subsection provides formal proofs for formulas that other people have worked out by observation. The second subsection discusses a triangle that is an original work by the authors of the paper.

2.1 Formulas by other scholars

Noah Carey and Greg Dresden [2] formulated various formulas regarding the sums of rows of triangles formed with Pascal's rule (1) but with different sets on the outer diagonals; however, those formulas lacked formal proofs. In this section, the usage of formula (2) in proving those formulas will be shown.

Triangle	Sets on two Diagonals	Formula
Pascal-Lucas	Lucas on both	$2(2^{n+1} - L_{n+1})$
Lucas-Counting	Lucas and Counting	$3(2^n) - L_{n+1} - 1$
Fibonacci-Counting	Fibonacci and Counting	$2^{n+1} - F_{n+1} - 1$
Tribonacci-One	Tribonacci and Ones	$5(2^{n-1}) + T_n - T_{n+3}$

Table 1: Formulas by Noah Carey and Greg Dresden

- ◊ Lucas numbers are numbers with base case $L_0 = 2, L_1 = 1$. They have the basic identity:

$$L_n = L_{n-1} + L_{n-2} \tag{3}$$

- ◊ Fibonacci numbers are numbers with base case $F_0 = 0, F_1 = 1$. They have the basic identity:

$$F_n = F_{n-1} + F_{n-2} \tag{4}$$

- ◊ Tribonacci numbers are numbers with base case $T_0 = 0, T_1 = 1, T_2 = 1$. They have the basic identity:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \tag{5}$$

- ◊ Counting numbers are the numbers $1, 2, 3, \ldots, n$

2.1.1 Pascal-Lucas

The Pascal-Lucas triangle contains Lucas numbers on both diagonals and appears in the OEIS [Online Encyclopedia of Integer Sequences] as the entry A347584.

$n = 0$							2						
$n = 1$						1		1					
$n = 2$					3		2		3				
$n = 3$				4		5		5		4			
$n = 4$			7		9		10		9		7		
$n = 5$		11		16		19		19		16		11	
$n = 6$	18		27		35		38		35		27		18

Table 2: First Rows of the Pascal-Lucas triangle

Theorem 2. *In Lucas-Pascal Triangle, for any $n \geq 0$, the sum of entries in the n-th row, $R(n)$, is calculated using this formula:*

$$R(n) = 2(2^{n+1} - L_{n+1}) \tag{6}$$

Proof. The proof will proceed by induction, where by substituting 0 in equation (6), we get 2. That is correct by observation. Now, we can assume, for the purpose of induction, that the equation holds for all n. Then, we can use the equation (2) with substituting all $T_{n,k}$ with the suitable Lucas. We get the following:

$$R(n+1) \ = \ 2 \cdot R(n) + 2 \cdot L_{n+1} - 2 \cdot L_n$$

Then by using the inductive hypothesis and rearranging terms:

$$R(n+1) \ = \ 2^{n+3} - 2 \cdot L_{n+1} - 2 \cdot L_n$$

Using the basic Lucas Identity (3) and rearranging terms, we get to the answer.

$$R(n+1) \ = \ 2(2^{n+2} - L_{n+2})$$

That is the result of substituting $n+1$ in the equation (6), which means that the induction holds. ■

2.1.2 Lucas-Counting

The Lucas-Counting triangle contains Lucas numbers on one diagonal and the counting numbers on the other.

$n = 0$							1						
$n = 1$						1		1					
$n = 2$					3		2		2				
$n = 3$				4		5		4		3			
$n = 4$			7		9		9		7		4		
$n = 5$		11		16		18		16		11		5	
$n = 6$	18		27		34		34		27		16		6

Table 3: First Rows of the Lucas-Counting triangle

Theorem 3. *In Lucas-Counting Triangle, for any $n \geq 0$, the sum of entries in the n-th row, $R(n)$, is calculated using this formula:*

$$R(n) = 3(2^n) - L_{n+1} - 1 \tag{7}$$

Proof. The proof will proceed by induction. By substituting 0 in equation (7), we get 1. That holds true by observation. Now, we can assume, for the purpose of induction, that the equation holds for all n. Then, we can use the equation (2), substituting all $T_{n,k}$ with the suitable Lucas and counting numbers. We get the following:

$$R(n+1) \;=\; 2 \cdot R(n) + L_{n+1} + n + 1 - n - L_n$$

Then by using the inductive hypothesis and rearranging terms:

$$R(n+1) \;=\; 3(2^{n+1}) - L_{n+1} - L_n - 1$$

Using the basic Lucas Identity (3) and rearranging terms, we get to the answer.

$$R(n+1) \;=\; 3(2^{n+1}) - L_{n+2} - 1$$

That is the result of substituting $n + 1$ in the equation (7), which means that the induction holds. ■

2.1.3 Fibonacci-Counting

The Fibonacci-Counting triangle contains the Fibonacci numbers on one diagonal and the counting numbers on the other.

$n = 0$							0							
$n = 1$						1		1						
$n = 2$					1		2		2					
$n = 3$				2		3		4		3				
$n = 4$			3		5		7		7		4			
$n = 5$		5		8		12		14		11		5		
$n = 6$	8		13		20		26		25		16		6	

Table 4: First Rows of the Fibonacci-Counting triangle

Theorem 4. *In Fibonacci-Counting Triangle, for any* $n \geq 0$*, the sum of entries in the* n*-th row,* $R(n)$*, is calculated using this formula:*

$$R(n) = 2^{n+1} - F_{n+1} - 1 \tag{8}$$

Proof. The proof will proceed by induction. By substituting 0 in equation (8), we get 0. That is correct by observation. Now, we can assume, for the purpose of induction, that the equation holds for all n. Then, we can use the equation (2) with substituting all $T_{n,k}$ with the suitable Fibonacci and counting numbers. We get the following:

$$R(n+1) \;=\; 2 \cdot R(n) + F_{n+1} + n + 1 - n - F_n$$

Then by using the inductive hypothesis and rearranging terms:

$$R(n+1) \;=\; 2^{n+2} - F_{n+1} - 1 - F_n$$

Using the basic Fibonacci Identity (4) and rearranging terms, we get to the answer.

$$R(n+1) \;=\; 2^{n+2} - F_{n+2} - 1$$

That is the result of substituting $n + 1$ in the equation (8), which means that the induction holds. ■

2.1.4 Tribonacci-One

The Tribonacci-One triangle contains the Tribonacci numbers on one diagonal and ones on the other.

$n = 0$							0						
$n = 1$						1		1					
$n = 2$					1		2		1				
$n = 3$				2		3		3		1			
$n = 4$			4		5		6		4		1		
$n = 5$		7		9		11		10		5		1	
$n = 6$	13		16		20		21		15		6		1

Table 5: First Rows of the Tribonacci-one triangle

Theorem 5. *In Tribonacci-One Triangle, for any $n \geq 1$, the sum of entries in the n-th row, $R(n)$, is calculated using this formula:*

$$R(n) = 5(2^{n-1}) + T_n - T_{n+3} \tag{9}$$

Proof. The proof will proceed by induction. By substituting 1 in equation (9), we get 2. That holds true by observation. Now, we can assume, for the purpose of induction, that the equation holds for all n. Then, we can use the equation (2), substituting all $T_{n,k}$ with the suitable Tribonacci numbers and ones. We get the following:

$$R(n+1) \;=\; 2 \cdot R(n) + T_{n+1} + 1 - 1 - T_n$$

Then by using the inductive hypothesis and rearranging terms:

$$R(n+1) \;=\; 5(2^n) + T_{n+1} - 2 \cdot T_{n+3} + T_n$$

Using the basic Tribonacci Identity (5) and rearranging terms, we get to the answer.

$$R(n+1) \;=\; 5(2^n) + T_{n+1} - T_{n+4}$$

That is the result of substituting $n + 1$ in the equation (9), which means that the induction holds. ■

2.2 Alternating Fibonacci-Lucas Triangle

The alternating Fibonacci-Lucas triangle has the sequence A005013 on both of its diagonals. The sequence has F_n for even n and L_n for odd n. It is necessary to know that there is a relation between F_n and L_n for all n.

$$L_n = F_{n-1} + F_{n+1} \tag{10}$$

See the alternating sequence and the first few rows of the triangle.

seq	0	1	2	3	4	5	6	7	8	9	10	11	12
Fib	0	1	1	2	3	5	8	13	21	34	55	89	144
Luc	2	1	3	4	7	11	18	29	47	76	123	199	322
A005013	0	1	1	4	3	11	8	29	21	76	55	199	144

Table 6: Formation of Sequence A005013

$n = 0$							0						
$n = 1$						1		1					
$n = 2$					1		2		1				
$n = 3$				4		3		3		4			
$n = 4$			3		7		6		7		3		
$n = 5$		11		10		13		13		10		11	
$n = 6$	8		21		23		26		23		11		8

Table 7: Rows of the alternating triangle

This time, the triangle has 2 formulas: one for even rows, and one for odd rows. That implies the necessity of proving each formula separately. Although we will need to prove that $R(n) \to R(n+2)$ for both even and odd n, we will use the previous technique in proofs. Before moving to the proofs, two lemmas should be proven.

Lemma 1. *For $n \geq 1$, the following holds:*

$$2 \cdot L_{n+4} + 2 \cdot L_{n+2} = 10 \cdot L_{n+1} + 10 \cdot F_{n-1} \tag{11}$$

Proof. The proof will proceed by stating that the R.H.S is equal to the L.H.S.

L.H.S $= 2 \cdot L_{n+4} + 2 \cdot L_{n+2}$, and using the formula (10), we get $2 \cdot F_{n+5} + 4 \cdot F_{n+3} + 2 \cdot F_{n+1}$. Now, we can repeatedly use equation (4) to get $10 \cdot F_{n+3}$.

R.H.S $= 10 \cdot L_{n+1} + 10 \cdot F_{n-1}$. By using the formula (10), we get $10 \cdot F_{n+2} + 10 \cdot F_n + 10 \cdot F_{n-1}$. Now, we can repeatedly use equation (4) to get $10 \cdot F_{n+3}$

Since R.H.S became equal to L.H.S, the lemma is true. ■

Lemma 2. *For $n \geq 1$, the following holds:*

$$6 \cdot L_{n+1} + 6 \cdot L_{n-1} + 20 \cdot L_n = 10 \cdot F_{n+1} + 10 \cdot L_{n+2} \tag{12}$$

Proof. The proof will proceed by stating that the R.H.S is equal to the L.H.S.

L.H.S $= 6 \cdot L_{n+1} + 6 \cdot L_{n-1} + 20 \cdot L_n$, where by using the equation (10), we can get the following formula: $6 \cdot F_{n+2} + 12 \cdot F_n + 6 \cdot F_{n-2} + 20 \cdot F_{n-1} + 20 \cdot F_{n+1}$. Here, using the basic the basic Fibonacci identity (4) in a repetitive manner gives the final shape: $40 \cdot F_{n+1} + 10 \cdot F_n$.

R.H.S $= 10 \cdot F_{n+1} + 10 \cdot L_{n+2}$, where we can easily use the formula (10) to get the Fibonacci-only form: $10 \cdot F_{n+1} + 10 \cdot F_{n+3} + 10 \cdot F_{n+1}$, and finally, the basic Fibonacci identity (4) gives us the final form: $40 \cdot F_{n+1} + 10 \cdot F_n$.

Since R.H.S became equal to L.H.S, the lemma is true. ■

Dividing both Lemma (1) and (2) by 5 gives us the following two equations:

$$\frac{2}{5} \cdot L_{n+4} + \frac{2}{5} \cdot L_{n+2} = 2 \cdot L_{n+1} + 2 \cdot F_{n-1} \tag{13}$$

$$\frac{6}{5} \cdot L_{n+1} + \frac{6}{5} \cdot L_{n-1} + 4 \cdot L_n = 2 \cdot F_{n+1} + 2 \cdot L_{n+2} \tag{14}$$

Theorem 6. *For all even $n \geq 0$, the sum of entries in the n-th row, $R(n)$ is:*

$$R(n) = \frac{14}{5} \cdot 2^n - \frac{2}{5} L_{n+4} \tag{15}$$

Proof. The proof will proceed by induction. First, by using 0 as a base case, we get 0 as $L_4 = 7$ from table (6). The base case clearly holds true by observing the triangle. Let's assume that the equation (15) is true for all even n for the purpose of induction. At first, we can use equation (2) to obtain the sum of entries in the $n+1-$th row. That gives us

$$R(n+1) \;=\; 2 \cdot R(n) + T_{n+1,0} + T_{n+1,n+1} - T_{n,n} - T_{n,0}$$

We can now apply the equation (2) again to get the formula of $R(n+2)$ in terms of $R(n)$

$$R(n+2) \;=\; 4 \cdot R(n) + T_{n+2,n+2} + T_{n+2,0} + T_{n+1,0} + T_{n+1,n+1} - 2T_{n,n} - 2T_{n,0}$$

We will replace all odd terms with L_n and all even ones with F_n (definition of the sequence (6)). Also, we should use the basic definition of Fibonacci: Equation (4), so that we can get the following:

$$R(n+2) \;=\; 4 \cdot R(n) + 2 \cdot F_{n-1} + 2 \cdot L_{n+1}$$

Replacing $R(n)$ with the inductive hypothesis, we reach

$$R(n+2) \;=\; \frac{14}{15} \cdot 2^{n+2} - \frac{2}{5} L_{n+6} - \frac{2}{5} \cdot L_{n+4} - \frac{2}{5} \cdot L_{n+2} + 2 \cdot F_{n-1} + 2 \cdot L_{n+1}$$

Using the equality relation of Lemma (1), we can get the final form:

$$R(n+2) \;=\; \frac{14}{15} \cdot 2^{n+2} - \frac{2}{5} L_{n+6}$$

That is the result of using equation (15) with $n+2$, where since the equation holds for the base case, and $R(n) \rightarrow R(n+2)$, the equation is accurate for all even n. ∎

Theorem 7. *For all odd $n \geq 1$, the sum of entries in the n-th row, $R(n)$ is:*

$$R(n) = \frac{14}{5} \cdot 2^n - \frac{6}{5} L_{n+1} \tag{16}$$

Proof. The proof will proceed by induction. First, by using 1 as a base case, we get 2 as $L_2 = 3$ from table (6). The base case clearly holds true by observing the triangle. Let's assume that the equation (16) is true for all odd n for the purpose of induction. At first, we can use equation (2) to get the sum of entries in the $n+1-$th row. That gives us

$$2 \cdot R(n) + T_{n+1,0} + T_{n+1,n+1} - T_{n,n} - T_{n,0}$$

We can again apply the equation (2) to get the formula of $R(n+2)$ in terms of $R(n)$

$$4 \cdot R(n) + T_{n+2,n+2} + T_{n+2,0} + T_{n+1,0} + T_{n+1,n+1} - 2T_{n,n} - 2T_{n,0}$$

Replacing all odd terms with L_n and even ones with F_n (definition of the sequence (6)), We observe the following:

$$4 \cdot R(n) + 2 \cdot L_{n+2} + 2 \cdot F_{n+1} - 4 \cdot L_n$$

Replacing $R(n)$ with the inductive hypothesis, we reach

$$\frac{14}{15} \cdot 2^{n+2} - \frac{6}{5} L_{n+3} - \frac{6}{5} \cdot L_{n+1} - \frac{6}{5} \cdot L_{n-1} + 2 \cdot L_{n+2} + 2 \cdot F_{n+1} - 4 \cdot L_n$$

Using the equality relation of Lemma (2), we can get the final form:

$$\frac{14}{15} \cdot 2^{n+2} - \frac{6}{5} L_{n+3}$$

That is the result of using equation (16) with $n+2$, where since the equation holds for the base case, and $R(n) \rightarrow R(n+2)$, the equation is accurate for all odd n. ■

3 K-Pascal Generalizations

Those Pascalized triangles are ones formed by using an arbitrary set on the diagonals (maybe 1s as Pascal did or other sets as those used in subsection (2.1); however, those generalizations are special because they use a formula other than formula (1), used by Pascal. Each subsection will discuss a certain k-pascal generalization, and the last subsection gives a generalized theorem for all k-pascal generalizations.

3.1 4-pascal Generalizations

In this generalization, each entry is the sum of the *four* entries above it, where an entry in the normal Pascal Triangle is the sum of the *two* entries above it. Moreover, $T_{n,k}$ is defined as being 0 for $k < 0$ or $k > n$.

$$T_{n,k} = T_{n-1,k-1} + T_{n-1,k} + T_{n-1,k+1} + T_{n-1,k+2} \tag{17}$$

The sum of entries of the n-th row in the 4-Pascal triangle can be achieved through the following recursive formula:

$$R(n) = 4 \cdot R(n-1) + T_{n,0} + T_{n,n} - 2 \cdot T_{n-1,0} - T_{n-1,1} - T_{n-1,n-2} - 2 \cdot T_{n-1,n-1} \tag{18}$$

Proof. The proof will proceed in a way similar to that of (2). First it is known that:

$$R(n) = \sum_{k=0}^{n} T_{n,k}$$

By using formula (17), we get:

$$R(n) = \sum_{k=0}^{n} T_{n-1,k} + \sum_{k=0}^{n} T_{n-1,k+1} + \sum_{k=0}^{n} T_{n-1,k-1} + \sum_{k=0}^{n} T_{n-1,k-2}$$

Since equation (17) is defined as being zero for both $k < 0$ and $k > n$, it is more accurate to write the previous equation as:

$$\begin{aligned} R(n) &= T_{n,0} + T_{n,1} + T_{n,n-1} + T_{n,n} \\ &\quad + \sum_{k=2}^{n-2} T_{n-1,k} + \sum_{k=2}^{n-2} T_{n-1,k+1} + \sum_{k=2}^{n-2} T_{n-1,k-1} + \sum_{k=2}^{n-2} T_{n-1,k-2} \end{aligned}$$

Writing the summations in terms of $R(n)$, we get:

$$\begin{aligned} R(n) &= T_{n,0} + T_{n,1} + T_{n,n-1} + T_{n,n} + \\ &\quad R(n-1) - T_{n-1,0} - T_{n-1,1} - T_{n-1,n-1} + \\ &\quad R(n-1) - T_{n-1,0} - T_{n-1,1} - T_{n-1,2} + \\ &\quad R(n-1) - T_{n-1,0} - T_{n-1,n-2} - T_{n-1,n-1} + \\ &\quad R(n-1) - T_{n-1,n-3} - T_{n-1,n-2} - T_{n-1,n-1} \end{aligned}$$

Grouping R(n-1)s together and substituting terms from the n-th row with equivalent terms from the $n-1$-th row whenever possible, we get the following:

$$\begin{aligned} R(n) &= T_{n,0} + T_{n,n} + 4R(n-1) \\ &\quad + T_{n-1,1} + T_{n-1,2} + T_{n-1,0} + T_{n-1,n-1} + T_{n-1,n-2} + T_{n-1,n-3} \\ &\quad - T_{n-1,0} - T_{n-1,1} - T_{n-1,n-1} \\ &\quad - T_{n-1,0} - T_{n-1,1} - T_{n-1,2} \\ &\quad - T_{n-1,0} - T_{n-1,n-2} - T_{n-1,n-1} \\ &\quad - T_{n-1,n-3} - T_{n-1,n-2} - T_{n-1,n-1} \end{aligned}$$

Removing equivalent terms with opposite signs and grouping the like ones, we get:

$$R(n) \;=\; T_{n,0} + T_{n,n} + 4R(n-1) - 2T_{n-1,0} - T_{n-1,1} - T_{n-1,n-2} - 2T_{n-1,n-1}$$

, which is the same as equation (18). ■

Now that we have proved the sum of rows of 4-pascal triangles, we can view two different generalizations and some facts about the sum of entries in their rows.

3.1.1 4-Pascal triangle

This triangle uses the equation (17), and if the terms of the equation are absent, we substitute them with 1.

$n=0$							1						
$n=1$						1		1					
$n=2$					2		2		2				
$n=3$				4		6		6		4			
$n=4$			10		16		20		16		10		
$n=5$		26		46		62		62		46		26	
$n=6$	72		134		196		216		196		134		72

Table 8: First few rows of the 4-pascal triangle

An interesting fact is that the sum of rows of this triangle can be abbreviated. According to formula (17), the following holds:

$$T_{n,n} = T_{n-1,n-1} + T_{n-1,n-2}, T_{n,0} = T_{n-1,0} + T_{n-1,1}$$

Substituting that in formula (18), we get:

$$R(n) = 4 \cdot R(n-1) - T_{n-1,0} - T_{n-1,n-1} \tag{19}$$

3.1.2 All-Ones 4-Pascal triangle

This triangle is formed the same way as the triangle (8); however, diagonals are restricted to always have 1's on the outer diagonals. That triangle appears on the encyclopedia as the entry A356692

$n=0$							1						
$n=1$						1		1					
$n=2$					1		2		1				
$n=3$				1		4		4		1			
$n=4$			1		9		10		9		1		
$n=5$		1		20		29		29		20		1	
$n=6$	1		50		79		98		79		50		1

Table 9: First few rows of the all-ones 4-pascal triangle

An interesting fact is that the sum of rows of this triangle (Equation (18)) can be abbreviated as $T_{n,n} = T_{n,0} = T_{n-1,n-1} = T_{n-1,0} = 1$ to be

$$R(n) = 4 \cdot R(n-1) - T_{n-1,0} - T_{n-1,1} - T_{n-1,n-2} - T_{n-1,n-1} \tag{20}$$

3.2 6-Pascal triangle

Analogous to the 4-Pascal triangle, each entry in the 6-Pascal triangle is the sum of the *six* entries above it. More precisely, the following equation holds whenever $k >= 0$ and $k <= n$. Otherwise, the entry is considered to be 0.

$$T_{n,k} = T_{n-1,k} + T_{n-1,k+1} + T_{n-1,k+2} + T_{n-1,k-1} + T_{n-1,k-2} + T_{n-1,k-3} \quad (21)$$

Since it is similar to both the normal Pascal triangle and the 4-Pascal triangle, it is expected to find a recursive formula for the sum of entries in the n-th row.

$$\begin{aligned} R(n) &= 6R(n-1) + T_{n,0} + T_{n,n} \\ &- 3T_{n-1,0} - 2T_{n-1,1} - T_{n-1,2} \\ &- 3T_{n-1,n-1} - 2T_{n-1,n-2} - T_{n-1,n-3} \end{aligned} \quad (22)$$

Proof. The proof will proceed in a similar way to that of both (2) and (18). First it's known that:

$$R(n) = \sum_{k=0}^{n} T_{n,k}$$

By using formula (21), we get:

$$\begin{aligned} R(n) = &\sum_{k=0}^{n} T_{n-1,k} + \sum_{k=0}^{n} T_{n-1,k+1} + \sum_{k=0}^{n} T_{n-1,k+2} + \\ &\sum_{k=0}^{n} T_{n-1,k-1} + \sum_{k=0}^{n} T_{n-1,k-2} + \sum_{k=0}^{n} T_{n-1,k-3} \end{aligned}$$

Since equation (21) is defined as being zero for both $k < 0$ and $k > n$, it is more accurate to write the previous equation as:

$$\begin{aligned} R(n) = &T_{n,0} + T_{n,1} + T_{n,2} + T_{n,n-2} + T_{n,n-1} + T_{n,n} + \\ &\sum_{k=3}^{n-3} T_{n-1,k} + \sum_{k=3}^{n-3} T_{n-1,k+1} + \sum_{k=3}^{n-3} T_{n-1,k+2} + \\ &\sum_{k=3}^{n-3} T_{n-1,k-1} + \sum_{k=3}^{n-3} T_{n-1,k-2} + \sum_{k=3}^{n-3} T_{n-1,k-3} \end{aligned}$$

Writing the summations in terms of $R(n)$, we get:

$$\begin{aligned} R(n) = &T_{n,0} + T_{n,1} + T_{n,2} + T_{n,n-2} + T_{n,n-1} + T_{n,n} + \\ &R(n-1) - T_{n-1,0} - T_{n-1,1} - T_{n-1,2} - T_{n-1,n-2} - T_{n-1,n-1} + \\ &R(n-1) - T_{n-1,0} - T_{n-1,1} - T_{n-1,2} - T_{n-1,3} - T_{n-1,n-1} + \\ &R(n-1) - T_{n-1,0} - T_{n-1,1} - T_{n-1,2} - T_{n-1,3} - T_{n-1,4} + \\ &R(n-1) - T_{n-1,0} - T_{n-1,1} - T_{n-1,n-3} - T_{n-1,n-2} - T_{n-1,n-1} + \\ &R(n-1) - T_{n-1,0} - T_{n-1,n-4} - T_{n-1,n-3} - T_{n-1,n-2} - T_{n-1,n-1} + \\ &R(n-1) - T_{n-1,n-5} - T_{n-1,n-4} - T_{n-1,n-3} - T_{n-1,n-2} - T_{n-1,n-1} \end{aligned}$$

Grouping R(n-1)s together and substituting terms from the n-th row with equivalent terms from the $n-1$-th row whenever possible, we get the following:

$$\begin{aligned}
R(n) &= T_{n,0} + T_{n,n} + 6R(n-1) \\
&+ T_{n-1,1} + T_{n-1,0} + T_{n-1,2} + T_{n-1,3} \\
&+ T_{n-1,2} + T_{n-1,1} + T_{n-1,0} + T_{n-1,3} + T_{n-1,4} \\
&+ T_{n-1,n-2} + T_{n-1,n-1} + T_{n-1,n-3} + T_{n-1,n-4} + T_{n-1,n-5} \\
&+ T_{n-1,n-1} + T_{n-1,n-2} + T_{n-1,n-3} + T_{n-1,n-4} \\
&- T_{n-1,0} - T_{n-1,1} - T_{n-1,2} - T_{n-1,n-2} - T_{n-1,n-1} \\
&- T_{n-1,0} - T_{n-1,1} - T_{n-1,2} - T_{n-1,3} - T_{n-1,n-1} \\
&- T_{n-1,0} - T_{n-1,1} - T_{n-1,2} - T_{n-1,3} - T_{n-1,4} \\
&- T_{n-1,0} - T_{n-1,1} - T_{n-1,n-3} - T_{n-1,n-2} - T_{n-1,n-1} \\
&- T_{n-1,0} - T_{n-1,n-4} - T_{n-1,n-3} - T_{n-1,n-2} - T_{n-1,n-1} \\
&- T_{n-1,n-5} - T_{n-1,n-4} - T_{n-1,n-3} - T_{n-1,n-2} - T_{n-1,n-1}
\end{aligned}$$

Removing equivalent terms with opposite signs and grouping the like ones, we get:

$$\begin{aligned}
R(n) &= T_{n,0} + T_{n,n} + 6R(n-1) \\
&- 3T_{n-1,0} - 2T_{n-1,1} - T_{n-1,2} \\
&- 3T_{n-1,n-1} 2T_{n-1,n-2} - T_{n-1,n-3}
\end{aligned}$$

, which is the same as equation (22). ■

3.3 8-Pascal-Triangle

Analogous to the 6-Pascal triangle, each entry in the 8-Pascal triangle is the sum of the *eight* entries above it. More precisely, the following equation holds whenever $k >= 0$ and $k <= n$. Otherwise, the entry is considered to be 0.

$$\begin{aligned} T_{n,k} =& T_{n-1,k} + T_{n-1,k+1} + T_{n-1,k+2} + T_{n-1,k+3} + \\ & T_{n-1,k-1} + T_{n-1,k-2} + T_{n-1,k-3} + T_{n-1,k-4} \end{aligned} \tag{23}$$

Since it's similar to the normal Pascal triangle, the 4-Pascal triangle, and the 6-pascal-triangle, it is expected to find a recursive formula for the sum of entries in the n-th row.

$$\begin{aligned} R(n) &= 8R(n-1) + T_{n,0} + T_{n,n} \\ &- 4T_{n-1,0} - 3T_{n-1,1} - 2T_{n-1,2} - T_{n-1,3} \\ &- 4T_{n-1,n-1} - 3T_{n-1,n-2} - 2T_{n-1,n-3} - T_{n-1,n-4} \end{aligned} \tag{24}$$

Proof. The proof will proceed in a way similar to that of both (2), (18), and (22). First it's known that:

$$R(n) = \sum_{k=0}^{n} T_{n,k}$$

By using formula (21), we get:

$$\begin{aligned} R(n) =& \sum_{k=0}^{n} T_{n-1,k} + \sum_{k=0}^{n} T_{n-1,k+1} + \sum_{k=0}^{n} T_{n-1,k+2} + \sum_{k=0}^{n} T_{n-1,k+3} + \\ & \sum_{k=0}^{n} T_{n-1,k-1} + \sum_{k=0}^{n} T_{n-1,k-2} + \sum_{k=0}^{n} T_{n-1,k-3} + \sum_{k=0}^{n} T_{n-1,k-4} \end{aligned}$$

Since equation (23) is defined as being zero for both $k < 0$ and $k > n$, it is more accurate to write the previous equation as:

$$\begin{aligned} R(n) =& T_{n,0} + T_{n,1} + T_{n,2} + T_{n,3} + \\ & T_{n,n-3} + T_{n,n-2} + T_{n,n-1} + T_{n,n} + \\ & \sum_{k=4}^{n-4} T_{n-1,k} + \sum_{k=4}^{n-4} T_{n-1,k+1} + \sum_{k=4}^{n-4} T_{n-1,k+2} + \sum_{k=4}^{n-4} T_{n-1,k+3} + \\ & \sum_{k=4}^{n-4} T_{n-1,k-1} + \sum_{k=4}^{n-4} T_{n-1,k-2} + \sum_{k=4}^{n-4} T_{n-1,k-3} + \sum_{k=4}^{n-4} T_{n-1,k-4} \end{aligned}$$

Writing the summations in terms of $R(n)$, grouping R(n-1)s together, and substituting terms from the n-th row with equivalent terms from the $n-1$-th row whenever possible (similar to the proof of formula (22), we can get formula (24). ■

3.4 Generalization for any k-pascal triangle

Noticing the similarity between formulas (2), (18), (22), and (24), the authors can obtain a theorem.

Theorem 8. *For all even k, where k is the number of elements of the $n-1$-th row that contribute to each single entry in the n-th row, the sum of entries in the n-th row can be concluded recursively using the following equation:*

$$R(n) \;=\; k \cdot R(n-1) + T_{n,0} + T_{n,n} - \sum_{i=1}^{k/2} T_{n-1,k+i-1} - \sum_{i=1}^{k/2} T_{n-1,n-i} \tag{25}$$

There is no formal proof for this equation, but the authors of this paper *do* encourage further research on ways to prove this formula.

4 Conclusion

The main formulas of sum of rows of Pascal's Triangle could be feasibly generalized into other formulas, according to the type of the generalization. Many interesting formulas like those provided in the research paper could be generated by creatively fluctuating the sets on the diagonals or the ways of getting the k-th entry of the n-th row.

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